The Magic of the Complex Numbers

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Rationals

Students learn to solve the following equations in different ways.

- $3x = 15$
- $3x = 16$

Expanding Number Systems

The formal way to add rational numbers $\frac{16}{3} + \frac{5}{8} = \frac{16 \cdot 8 + 5 \cdot 3}{3 \cdot 8}$ is ideal for two reasons:

- It is how one adds quantities.
- It preserves the basic properties of the integers.

Basic Properties

- $a + b = b + a$
- $ab = ba$
- $a + (b + c) = (a + b) + c$
- $a(bc) = (ab)c$
- $a(b + c) = ab + ac$

Examples

The identity $(-1)(-1) = 1$ confused the great Euler.

More Examples

- $5^0 = 1$
- $5^{\frac{3}{2}} = (\sqrt{5})^3.$

Real Numbers

Students learn to solve the following equations in different ways.

- $x^2 = \frac{9}{25}$
- $x^2 = 2$
- $x^2 = -4.$

Expanding Number Systems (Again)

Expressions can become complicated

- $\sqrt{2} + \sqrt{3} = 1.4142\ldots + 1.7320\ldots$
- $17^{\sqrt{2}}$
- e^{i}.

Complex Numbers

In the 16th century mathematicians introduced expressions for the square roots of negative numbers like $\sqrt{-1} = i$ but regarded these numbers as useless or ”imaginary” at first. Nevertheless operations like multiplication

$$(a + bi)(c + di) = ac - bd + (ad + cb)i$$

could be worked out by combining the property $i \cdot i = -1$ with the old properties.

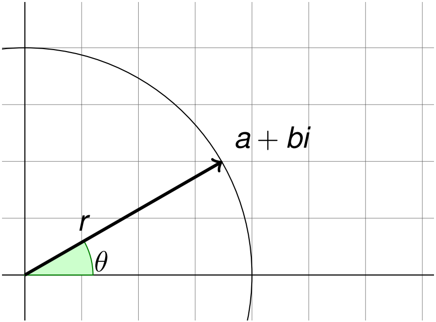
Examples

- $(2+i)^2$
- $\frac{2+i}{1-3i}$

Complex Numbers

In the 19th century complex numbers were found to have interesting geometric properties and were thus investigated seriously for the first time by mathematicians like the great Gauss.

The Geometry of Complex Numbers



Sum, Conjugate and Modulus

- the sum of complex numbers $z = a + bi$ and $w = c + di$ is given by the parallelogram law for adding vectors.
- the *conjugate* of $z = a + bi$ is $\bar{z} = a - bi$
- the *modulus* of a complex number $z = a + bi$ is $r = |z| = \sqrt{z\bar{z}}$.

Euler's Formula

- polar representation
  $z = r\cos\theta + ri\sin\theta = r(\cos\theta + i\sin\theta)$

- Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$

Properties of Exponentials

- laws of exponents $e^{a+bi} = e^a e^{ib} = re^{i\theta}$
- complex multiplication $r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$
- $\frac{d}{d\theta} e^{i\theta} = ie^{i\theta}$.

Applications

- Express $(1+i)^6$ in the form $a+bi$
- Express $\cos^3 x$ in terms of $\cos nx$ for suitable n.
- Evaluate $\int e^{-x} \cos x \mathrm{d}x$

Complex Functions

Complex functions $w = f(z)$ can also be studied. Here are some geometric examples

- Translation: $f(z) = z + (3 - i)$
- Rotation: $f(z) = (\frac{\sqrt{3}}{2} + \frac{1}{2}i)z$
- Mirror Reflection: $f(z) = \bar{z}$

Euclidean Geometry

Felix Klein defined *geometric properties* as those properties that are common to all congruent figures.

More Complex Functions

- $f(z) = z^2$
- $f(z) = \frac{1}{\bar{z}}$

More Examples

- Sketch the image of the circle $|z-1| = 1$ under the mapping $f(z) = z^2$.
- Show that the image of the perpendicular bisector of z_1 and z_2 is taken to the unit circle by the map $f(z) = \frac{z-z_1}{z-z_2}$.

Mobius Transformations

A *Mobius Transformation* is a complex function of the form $M(z) = \frac{az+b}{cz+d}$. These maps have the remarkable properties that

1. Mobius transformations map circles and lines to circles and lines.
2. Mobius transformations preserve angles.

Stereographic Projection

Stereographic projection combines ideas from geometry, topology, and complex numbers