Guarding Polygonal Museums

Joseph Malkevitch

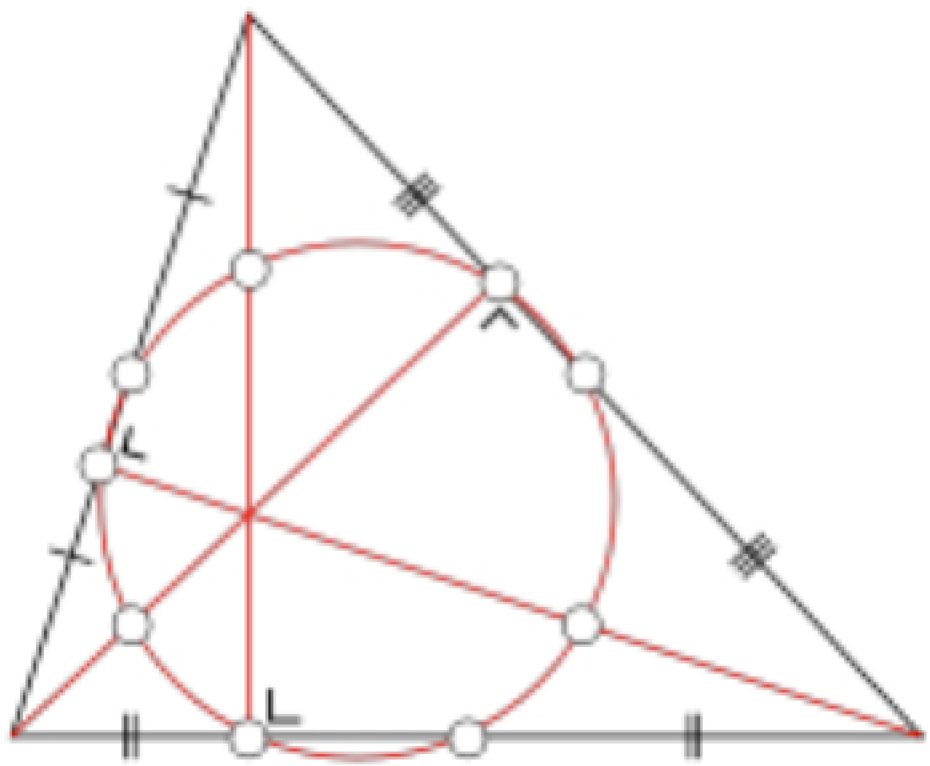
Department of Mathematics
York College(CUNY)

email:
jmalkevitch@york.cuny.edu

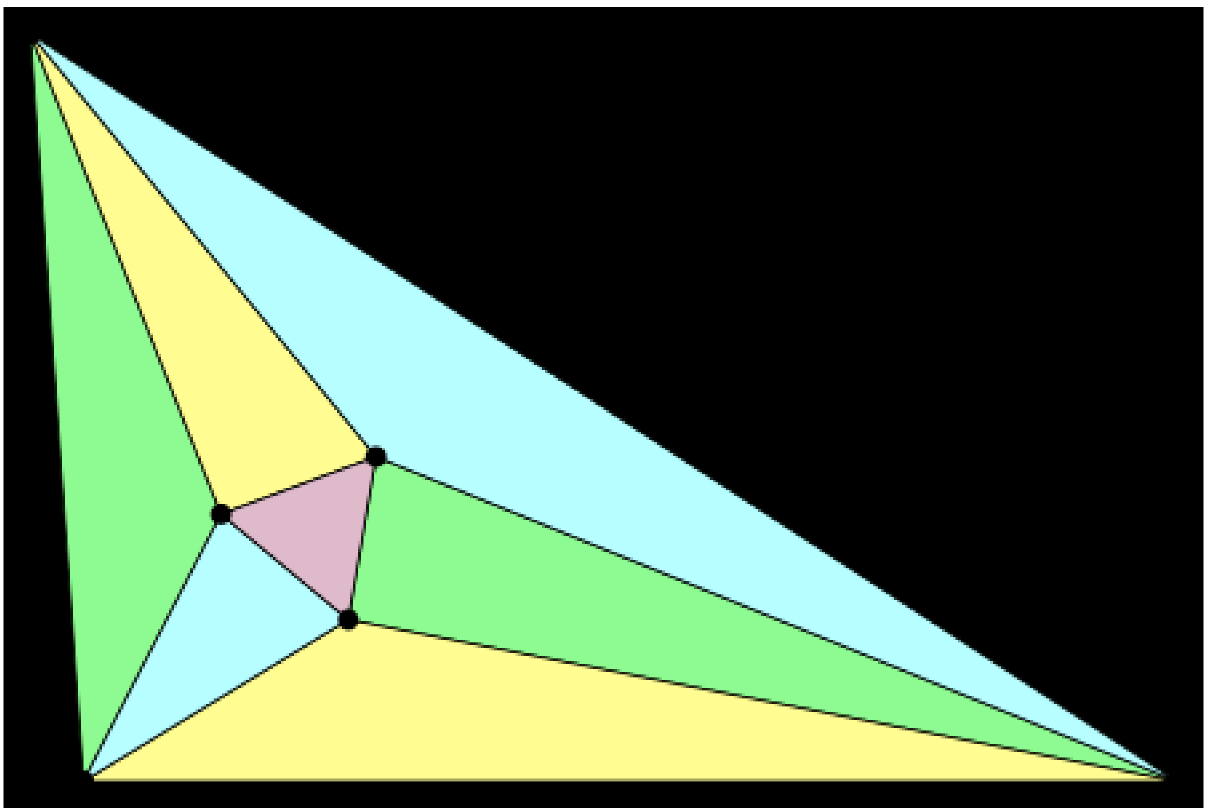
web page:
http://www.york.cuny.edu/~malk

Some spectacular theorems in Euclidean geometry:

Nine point Circle (1842):



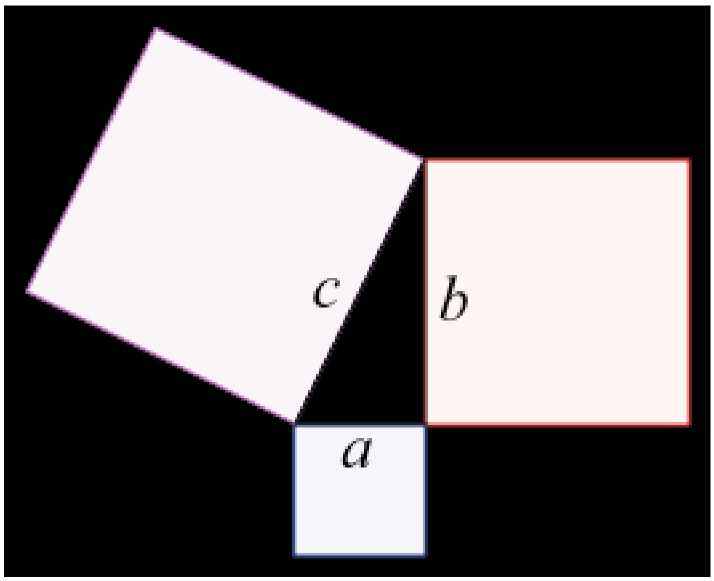
Frank Morley's Theorem:



We have angle trisectors at each vertex.

The "central" triangle is ***equilateral***!

Pythagorean Theorem:



The area of the squares on sides a and b of a right triangle add to the area of the square on side c.

Euler's Polyhedra Formula for Convex Polyhedra (1752):

$$V + F - E = 2$$

V = vertices

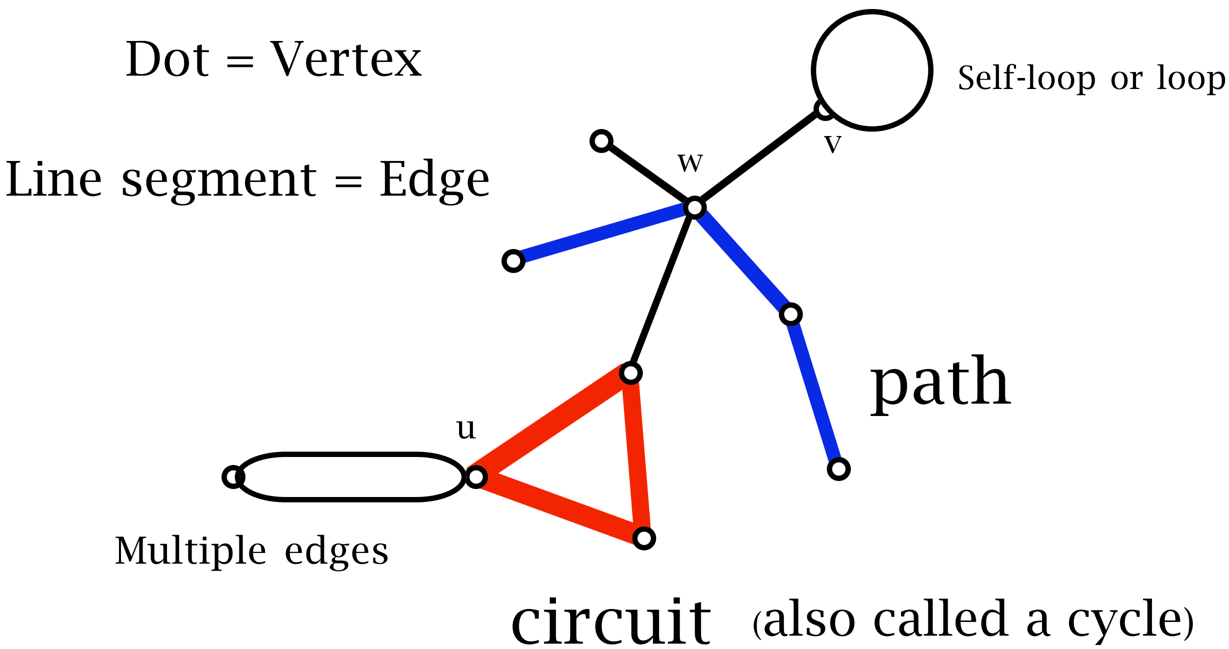
F = faces

E = edges

Steinitz's Theorem:

(reformulated by Branko Grünbaum and Theodore Motzkin)

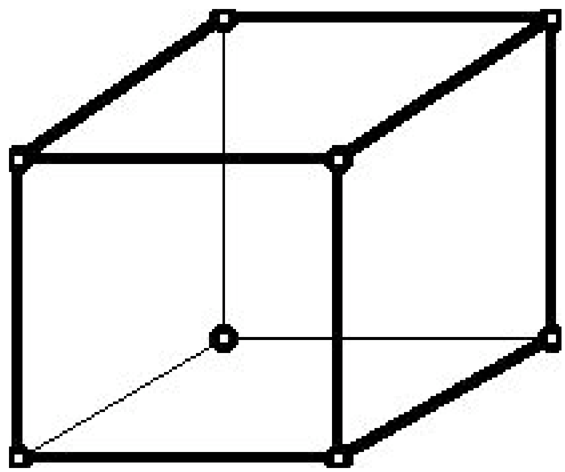
G is the vertex-edge graph of a convex 3-dimensional polytope if and only G is planar and 3-connected

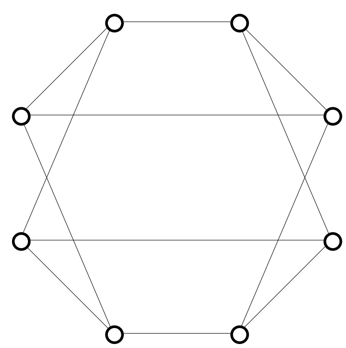


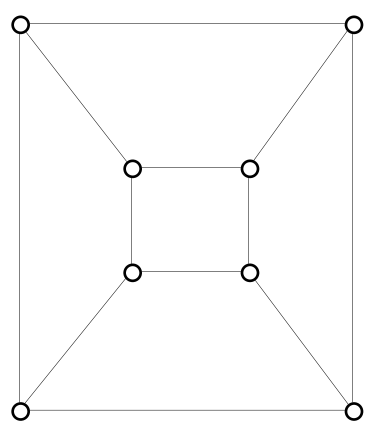
This graph has 10 vertices and 12 edges.

The valence or degree of a vertex in a graph is the number of (local) line segments which meet at the vertex. The valence of v is 3, of w is 5, and of u is 4.

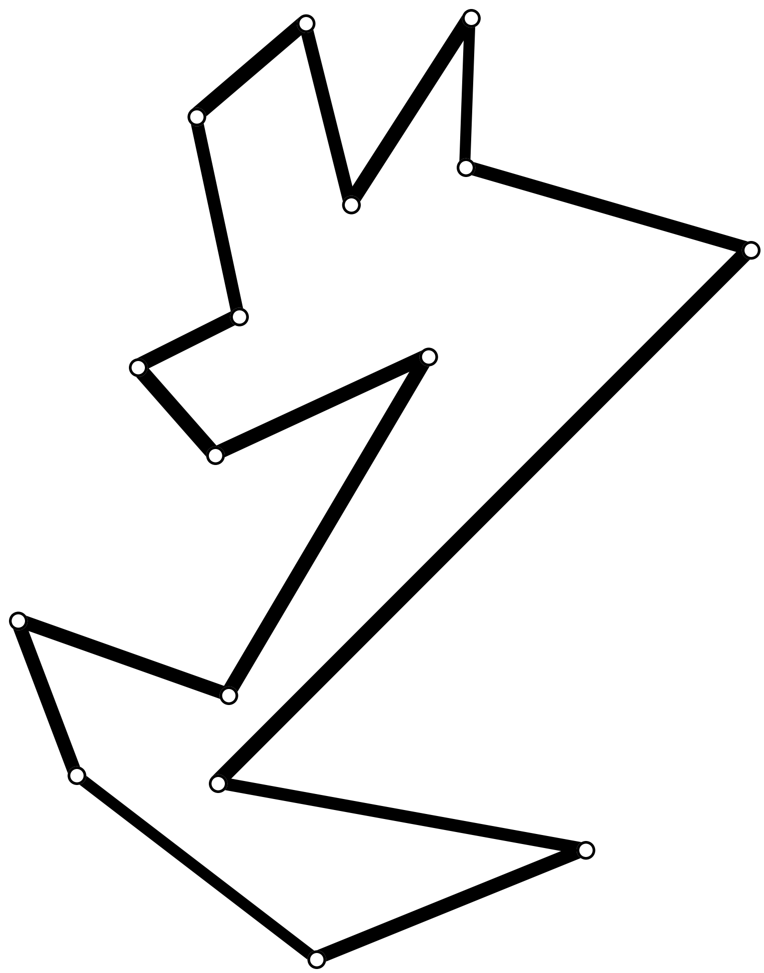
Combinatorial cube:



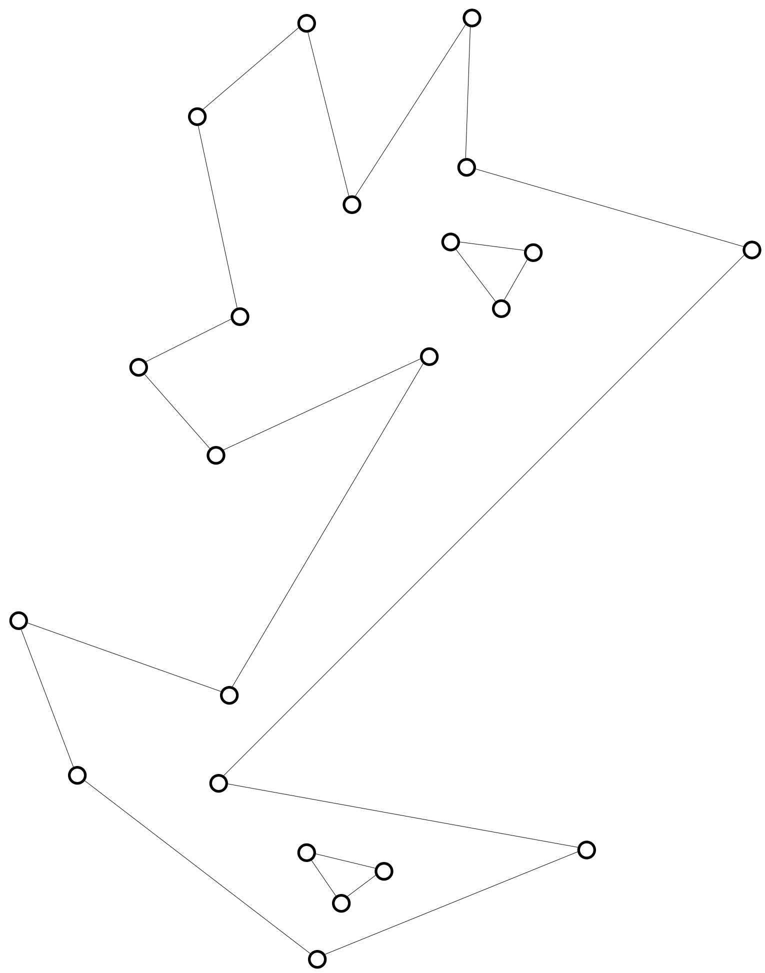




Museum floor plan = plane polygon



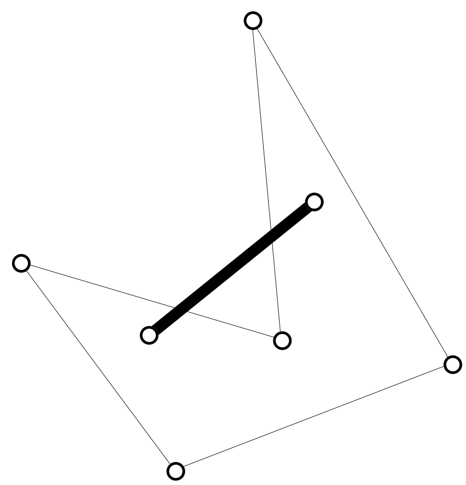
Museum with pillars:



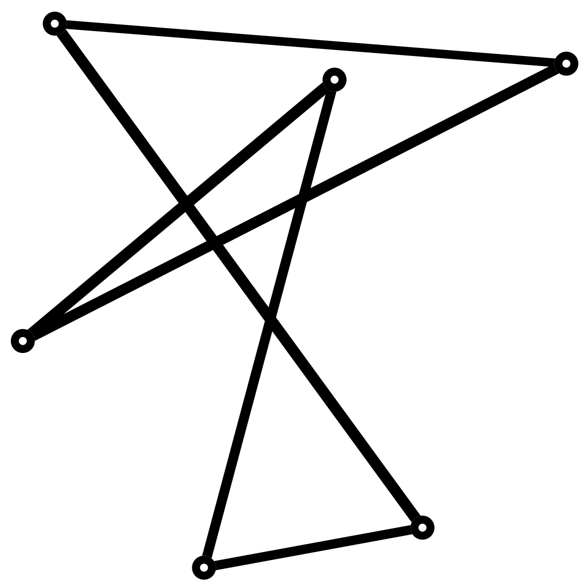
Not a polygon!

Types of polygons:

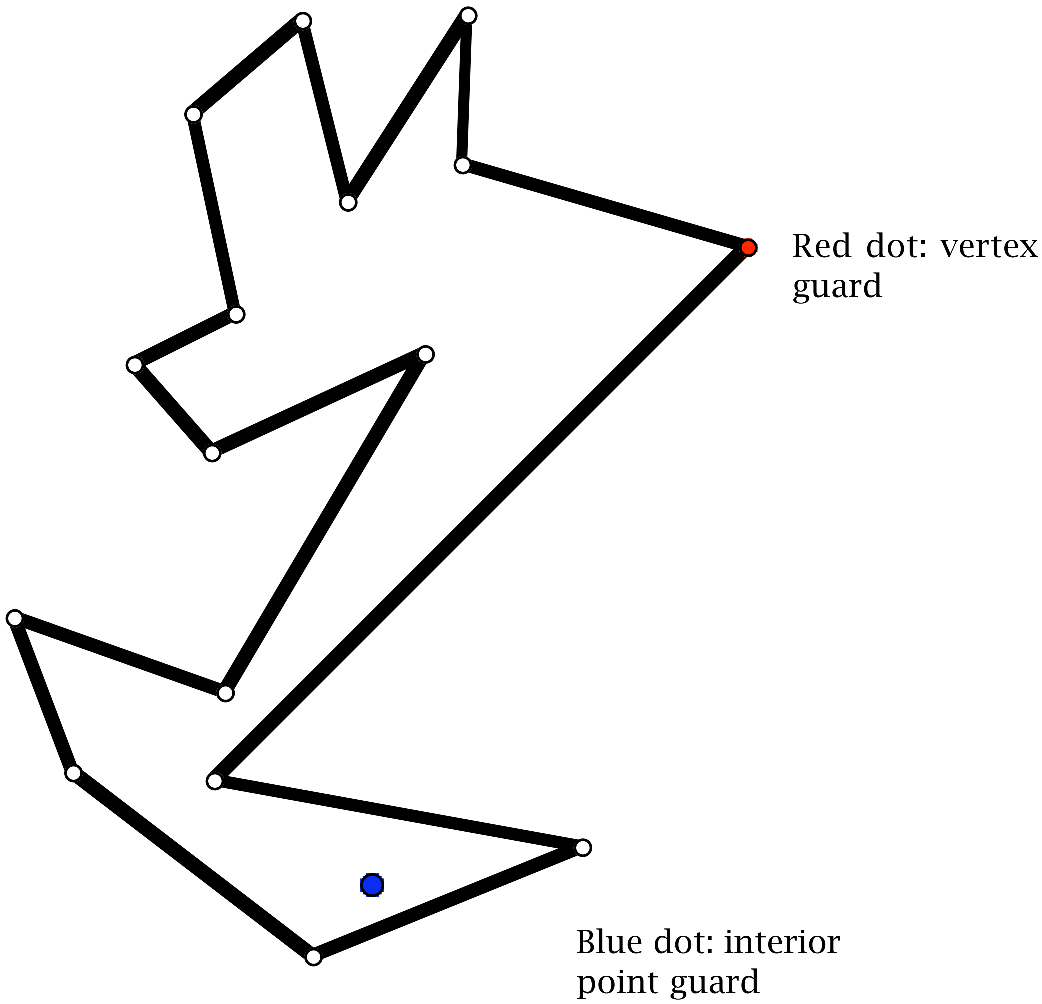
Non-convex:



Non-simple polygon:



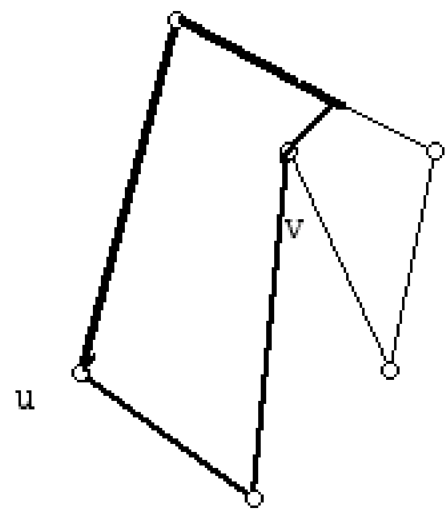
Surveillance devices = "guards"



Visibility:

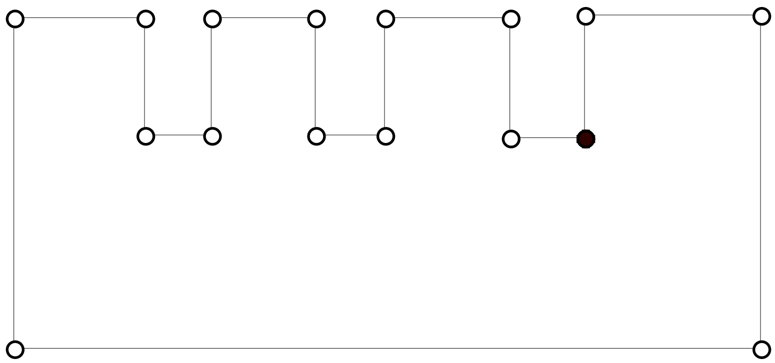
Point v in the interior of simple polygon P is visible by guard g (in the interior or boundary of P) if the line segment vg contains no point in the exterior of P.

Visibility polygon:



Visibility polygon from a vertex u is typically a non-convex polygon.

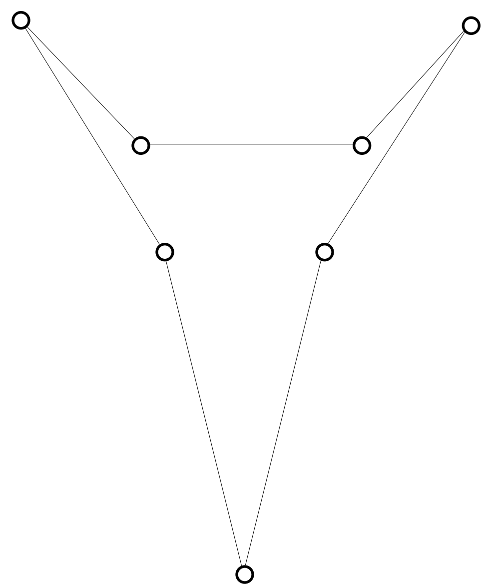
What can a guard "see?"



What can the guard at the black dot see?

Vision versus clear vision!

Vertex guards vs: interior point guards:



No vertex guard can see the whole polygon. There are interior point guards that can see the whole polygon.

Victor Klee (posed 1973)

Given an n sided simple plane polygon P how many "vertex guards" are sometimes necessary and always sufficient to guard P?

Note:

This is not the same problem as given a simple plane polygon P with n sides determine the minimum number of vertex guards to see all of P.

This problem is known to be NP-complete!

First proof (1975):

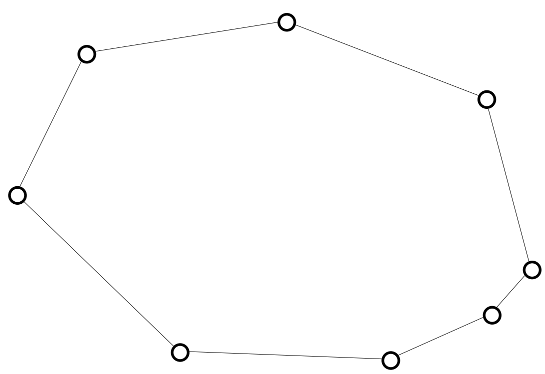
Vasek Chvátal

"Book" proof (1978):

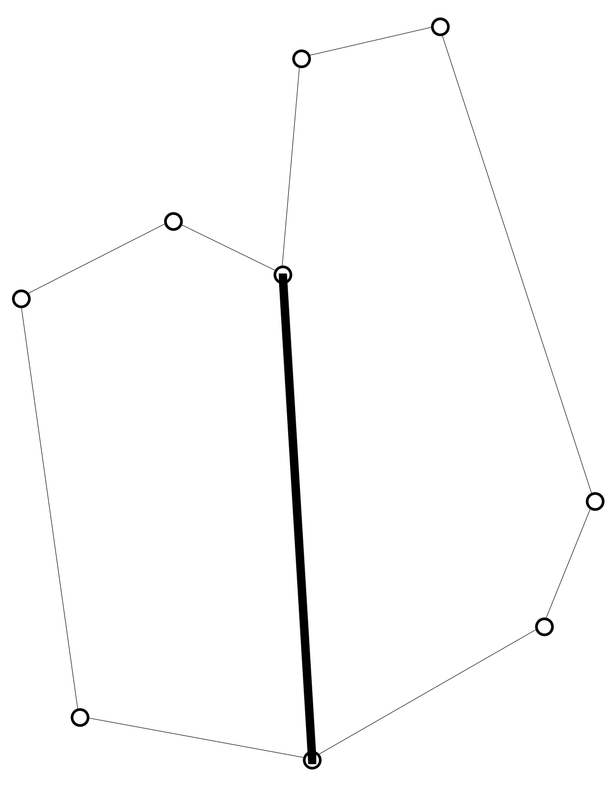
Steve Fisk.

Finding the right conjecture:

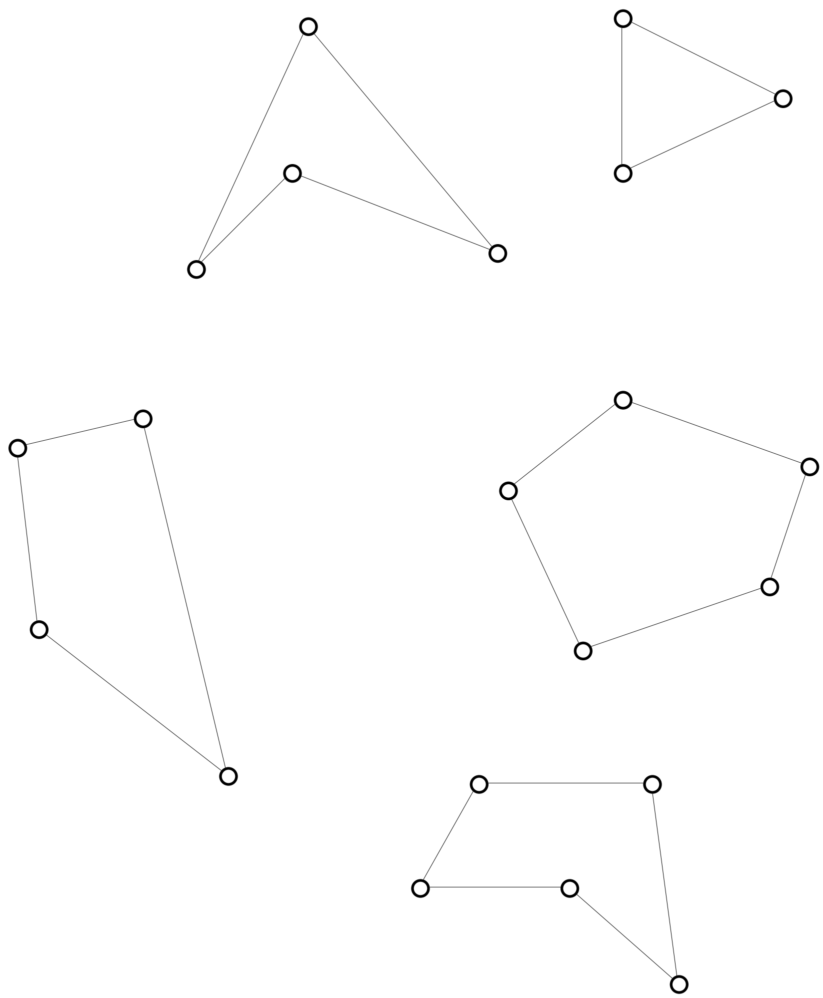
a. Convex polygons require only one guard for any number of sides:



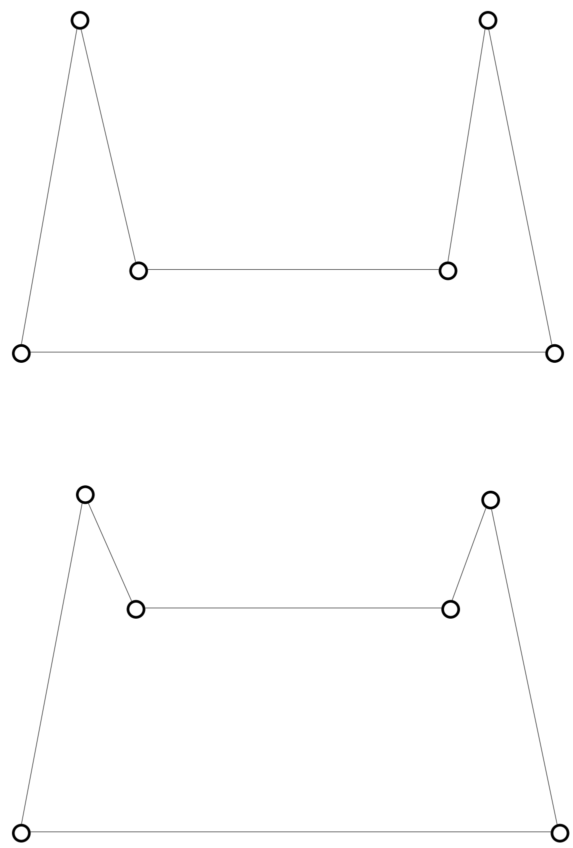
b. Polygons which can be partitioned by diagonals at a vertex into the union of convex polygons only require one guard:



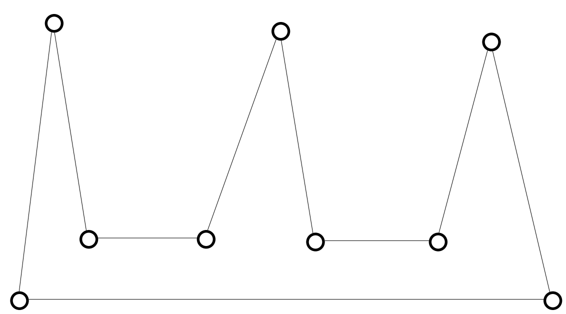
c. Polygons with few sides:



d. One can find a family of polygons which require more and more guards:



Adding three more sides requires one more guard:

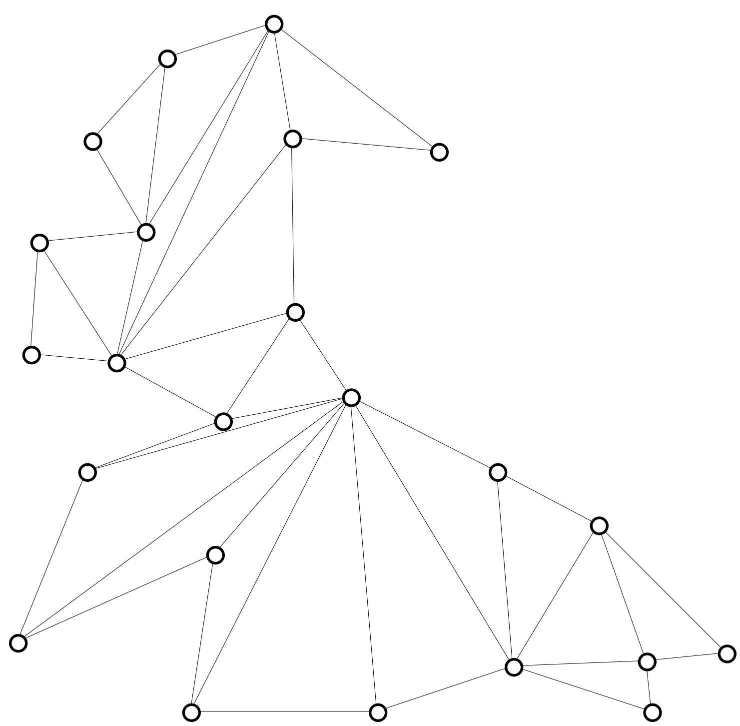


Hence:

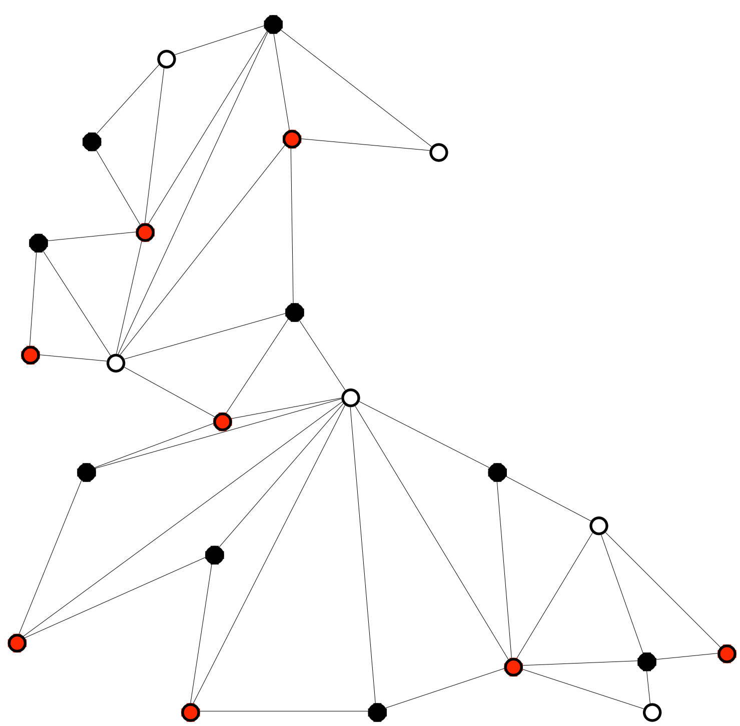
$\lfloor n/3 \rfloor$ guards are sometimes necessary!

Fisk's Proof:

a. Triangulate the given polygon

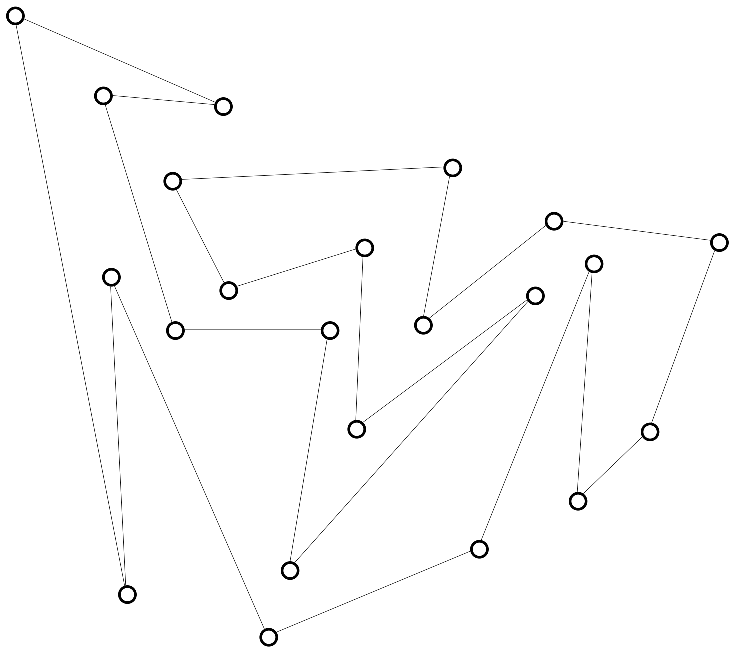


b. Three-color the vertices of the triangulated polygon



c. Put guards at the vertices which correspond to the color used least often!

Can every simple plane polygon be triangulated?

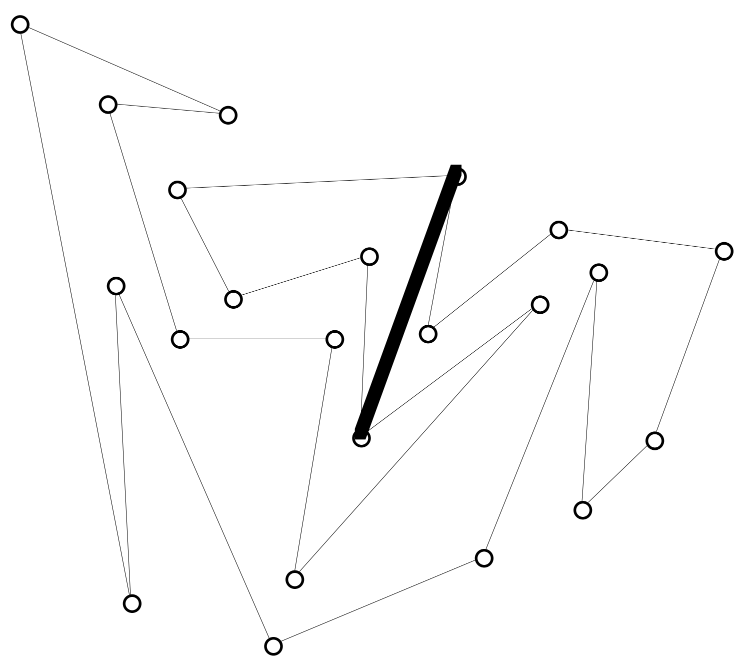


Can every 3-dimensional polyhedron topologically equivalent to a sphere be "tetrahedralized?"

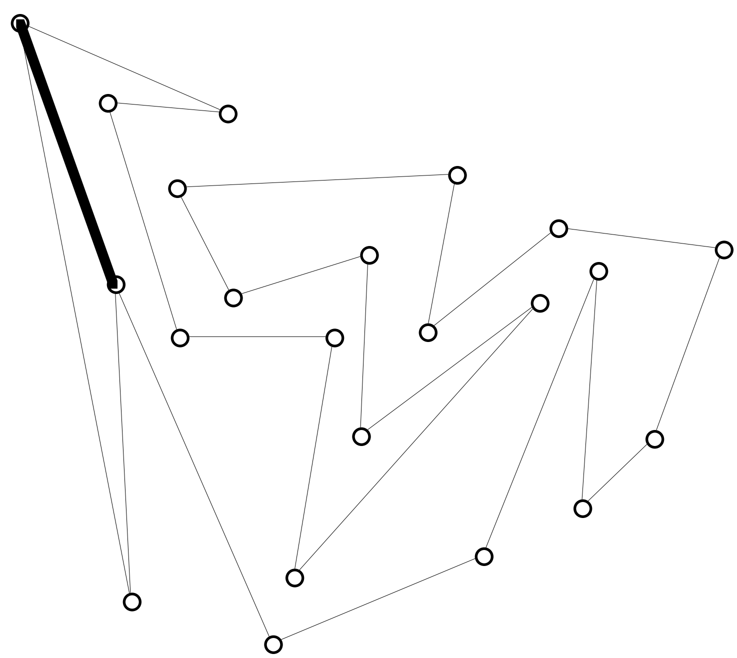
It is not the case that every 3-dimensional polyhedron can subdivided into simplices.

In fact, there exist 3-dimensional non-convex polyhedra where the line segments joining every pair of vertices not already joined lie in the polyhedron's exterior!

Polygon subdivided with a diagonal:



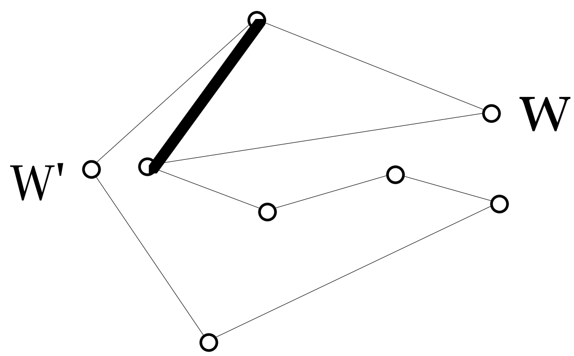
Polygon with an "ear" cut off:

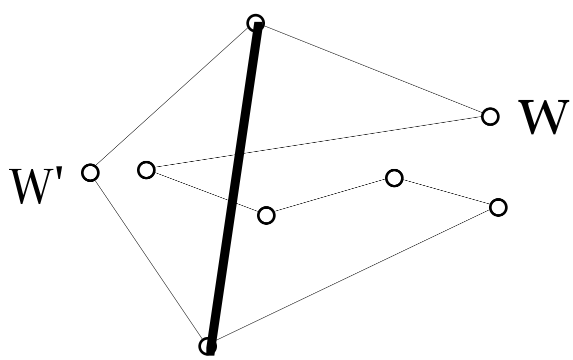


Definition:

An ***ear*** of a plane simple polygon is a vertex w so that the edge joining the vertices whose edges meet at w is an interior diagonal of the polygon.

w is an ear while w' is not an ear:



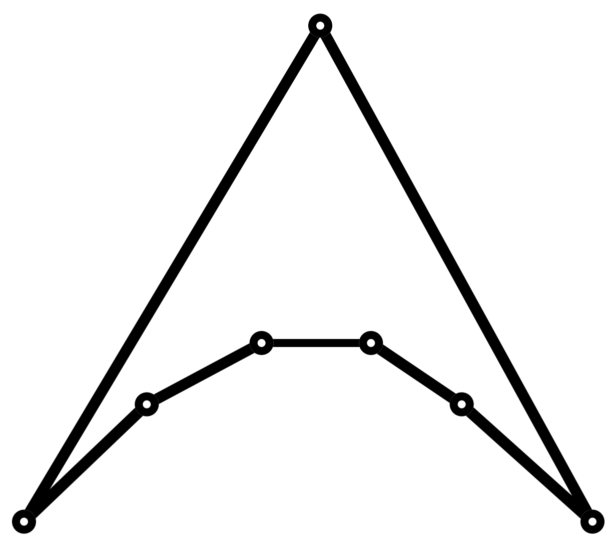


Gary Meister's Theorem:

Every plane simple polygon has at least two vertices which are ears and these ears determine triangles whose interiors do not overlap.

(Every triangulated polygon has at least two vertices of valence 2.)

There exists an infinite family of plane simple polygons with exactly two ears:



Variants:

a. Attempts to generalize the art gallery/museum problem to special types of polygons and higher dimensions.

b. Attempts to put the Fisk proof to work.

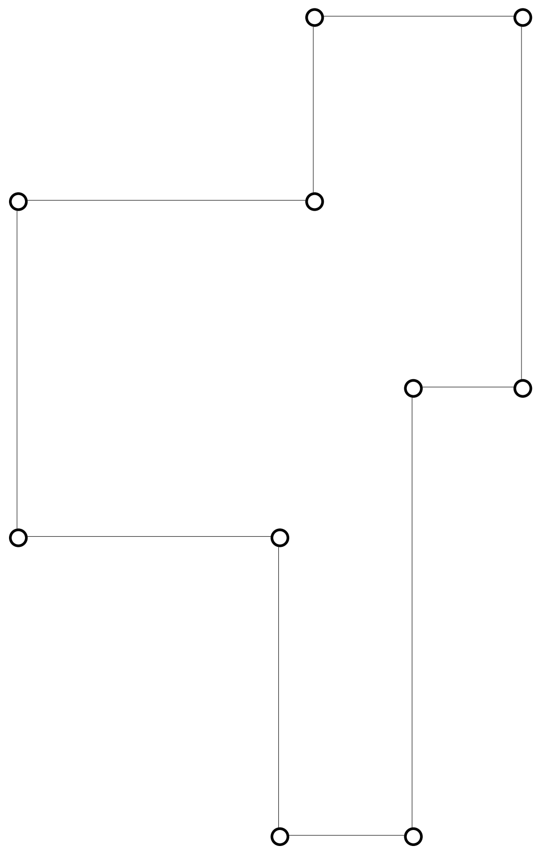
Theorem (1983):

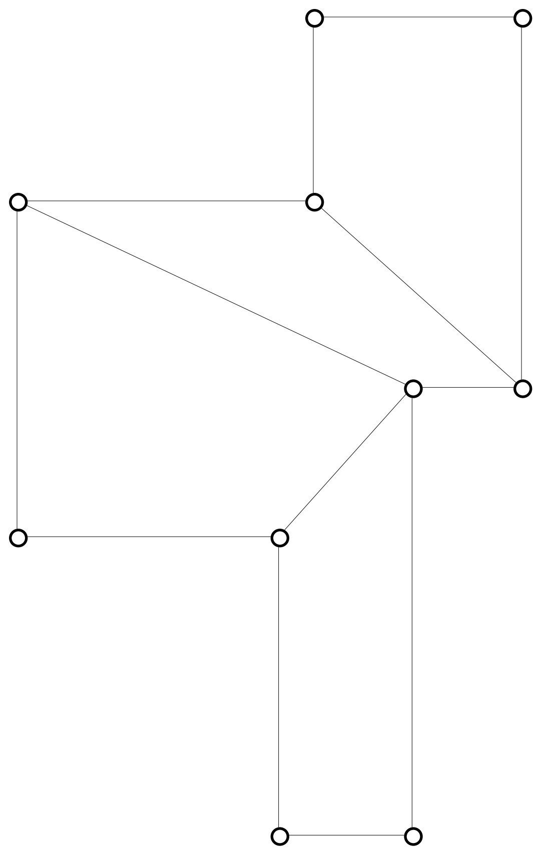
(Kahn, Klawe, Kleitman)

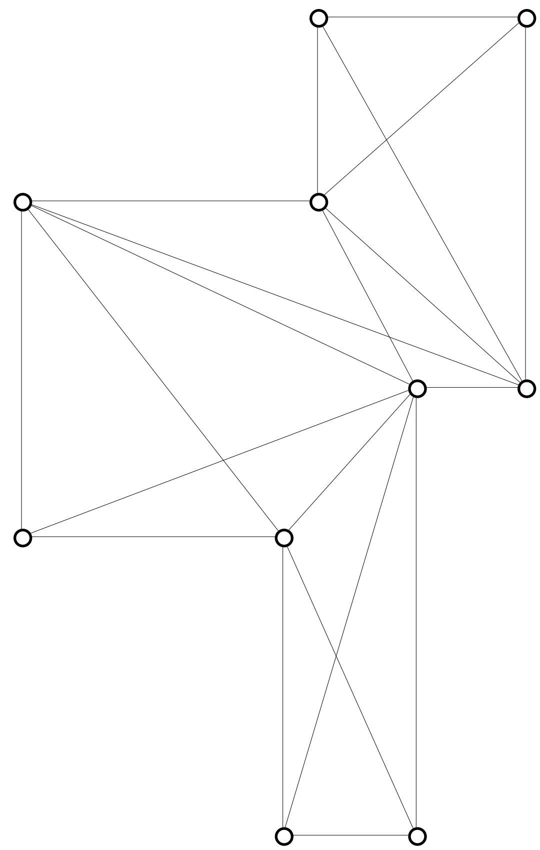
$\lfloor n/4 \rfloor$ guards are sometimes necessary and always sufficient to guard a plane simple orthogonal polygon

Lemma:

Every orthogonal simple plane polygon can be decomposed into convex quadrilaterals.



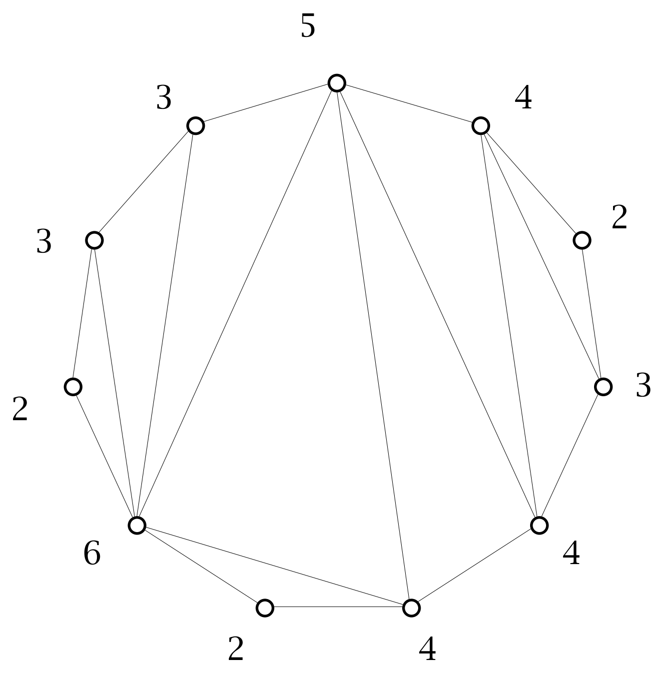




Now 4-color the graph above.

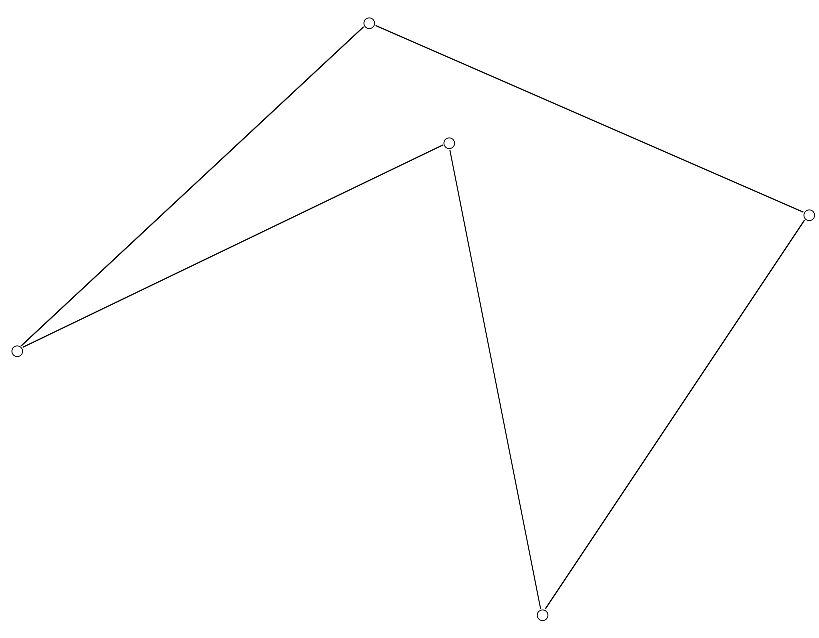
As yet unsolved problems:

Determine the integer sequences which can arises as valences of triangulated plane simple polygons:



Valence or degree sequence: 6, 5, 4, 4, 4, 3, 3, 3, 2, 2, 2

What is the number of vertex guards that are sometimes necessary and always sufficient for a plane simple ***equilateral*** n sided polygon?



equilateral 5-gon

Which plane 3-connected graphs admit even triangulations?

(Heawood's Theorem: A plane triangulation is vertex 3-colorable if and only if the triangulation is even-valent)

Mobile guards:

Guards who can move along edges or diagonals of a polygon.

Conjecture (G. Toussaint):

For large values of n, the number of edge guards

$$\lfloor n/4 \rfloor$$

are sometime necessary and always sufficient.

(Known: $\lfloor 3n/10 \rfloor$)

References:

1. O'Rourke, J. Art Gallery Theorems and Algorithms, Oxford, 1987.

2. Goodman J. E. and J. O'Rourke, (eds.),Handbook of Discrete and Computational Geometry, CRC, 2004.

3. Web searches:

(art gallery problems; floodlight problem; museum guard problem)